Bachelor of Arts (B.A.) Part—I Semester—II Examination

MATHEMATICS

(M_4 : Vector Calculus and Improper Integrals)

Optional Paper—2

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) A particle moves so that its position vector is given by $\bar{r} = \cos wt\,\bar{i} + \sin wt\,\bar{j}$ where w is a constant. Show that :

(a) the velocity $\bar{V}$ of the particle is perpendicular to $\bar{r}$.

(b) the acceleration $\bar{a}$ is directed towards the origin and has magnitude proportional to the distance from the origin.

(c) $\bar{r} \times \bar{v} = \bar{a}$, constant vector. 6

(B) If $\phi = x^3 + y^3 + z^3 - 3xyz$, then find :

(i) $\bar{\nabla} \circ (\bar{\nabla}\phi)$ and

(ii) $\bar{\nabla} \times (\bar{\nabla}\phi)$. 6

OR

(C) If $\bar{v} = \bar{w} \times \bar{r}$, then prove that $\bar{w} = \frac{1}{2}\,\text{curl}\,\bar{v}$, where $\bar{w} = w_1\bar{i} + w_2\bar{j} + w_3\bar{k}$ is a constant vector. 6

(D) If $\bar{A} = (3x^2 + 6y)\bar{i} - 14yz\,\bar{j} + 20xz^2\bar{k}$, then evaluate $\int_C \bar{A} \circ d\bar{r}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t,\ y = t^2,\ z = t^3$. 6

UNIT—II

2. (A) Evaluate $\iint_R xy\,dxdy$ over the region R in the positive quadrant for which $x + y \leq 1$. 6

(B) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dxdy$. 6

OR

(C) Evaluate $\iint_R (x^2+y)\,dxdy$ by changing into polar coordinates, where R is the region $x^2 + y^2 \leq 1$. 6

(D) Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dx\,dy$. 6

UNIT—III

3. (A) If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$. Then evaluate $\iiint_V \bar{F}\,dv$, where V is the region bounded by the surfaces $x = 0$, $y = 0$, $y = 6$, $z = x^2$, $z = 4$. 6

(B) Evaluate by Green's theorem $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region defined by $x = 0$, $y = 0$, $x + y = 1$. 6

OR

(C) Evaluate by Stoke's theorem $\int_C e^x dx + 2ydy - dz$, where C is the curve $x^2 + y^2 = 4$, $z = 2$. 6

(D) Evaluate $\iint_S \bar{F} \circ \hat{n}\,ds$, where $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ and S is the surface of the solid cut off by the plane $x + y + z = a$ from the first octant. 6

UNIT—IV

4. (A) Test the convergence of :

(i) $\int_0^{\infty} e^{-x^2} dx$ and

(ii) $\int_1^{\infty} \frac{\log x}{x+a}\,dx$ where a is a positive constant. 6

(B) Prove that if $\int_a^{\infty} |f(x)| dx$ converges, then $\int_a^{\infty} f(x) dx$ also converges. Hence prove that $\int_0^{\infty} \frac{\sin x}{x^2+1} dx$ is convergent. 6

OR

(C) Prove that $\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$. 6

(D) Prove that :

(i) $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$ and

(ii) $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$. 6

Questions—V

5\. (A) If $\phi = \log r$, then find $\bar{\nabla}\phi$, where $r = |\bar{r}|$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. 1½

(B) Show that $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ is solenoidal. 1½

(C) Change the order of $\int_0^2 \int_0^x f(x, y) dy \, dx$. 1½

(D) Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx$. 1½

(E) Apply Green's theorem to prove that the area enclosed by a simple plane curve C is $\frac{1}{2} \oint_C x dy - y dx$. 1½

(F) Find the area of ellipse $x = a \cos\theta$, $y = b \sin\theta$ by using Green's theorem. 1½

(G) Prove that $\Gamma 1 = 1$. 1½

(H) Evaluate $\beta\left(\frac{3}{2}, \frac{5}{2}\right)$. 1½